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ON THE STABILITY OF STATIONARY MOTIONS OF ROTATING ROTOR AXIS MOUNTED ON NONLINEAR BEARINGS*

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Stability of the steady motions of the axis of an unbalanced rotating rotor with perfect and imperfect motors is studied under the assmption that the reaction of elastically complaint bearings increases with increasing deformation. All basic parameters of the steady motions and their points of bifurcation are determined. It is shown that an imperfect motor can have a destabilizing influence. The results obtained are applicable to the case of a plane parallel motion of a rotating rotor on an isotropic, inertialess nonlinearly flexible shaft.

1. We consider a perfectly rigid rotor of mass m with vertical axis of rotation, mounted on elastically compliant bearings fixed rigidly on an immobile foundation. We assume that the rotor, the eccentricity of which is $e = OC$, moves in plane parallel manner, and the characteristic rotation about the axis O at constant frequency ω is executed by a perfect motor (motor of infinite power). The bearing reactions which are generally nonlinear, are reduced to the resultant $F_0(\rho)$ depending on the radial displacement $\rho = O_1O$ of the rotor axis and directed along the straight line OO_1 towards the point O_1 of intersection of the plane of motion of the center of mass C with the axis of the undeformed bearings /1- 5/. Physical considerations imply that any reaction $F_0(\rho)$ must vanish when $\rho = 0$ and increase with increasing ρ within the admissible limits of deformation of the bearings, i.e. the following conditions must hold:

$$F_0(0) = 0, \quad dF_0/d\rho > 0 \tag{1.1}$$

We also assume that the derivative $d^2F_0/d\rho^2$ is continuous within the same limits. The conditions can be satisfied by the rigid, as well as the soft elastic compliance of the bearings.

When the external reaction forces are absent, the differential equations of motion have the form /6/

$$\begin{aligned} &\rho^{\cdot\cdot} - \rho\varphi^{\cdot 2} - \omega^2 e\cos(\omega t - \varphi) = -F(\rho), \quad F(\rho) = F_0(\rho)/m \\ &\rho\varphi^{\cdot\cdot} + 2\rho^{\cdot}\varphi^{\cdot} - \omega^2 e\sin(\omega t - \varphi) = 0 \end{aligned} \tag{1.2}$$

where φ is the angle between the segment O_1O and the fixed x-axis. From the above equations we see that under the above assumptions all possible steady motions (cylindrical precessions)

$$\rho = r = \text{const}, \quad \varphi^{\cdot} = \text{const} \tag{1.3}$$

take place, in contrast to the balanced rotor $(e = 0)$, firstly only in the rectilinear direction, and secondly at a single frequency $\varphi^{\cdot}$ equal to the characteristic rotation frequency ω. The conclusion remains valid for any reaction forces depending on $\rho, \rho^{\cdot}$ and $\varphi^{\cdot}$. The constants r and $\gamma = \omega t - \varphi$ satisfy the equations

$$r\omega^2 + e\omega^2\cos\gamma = F(r), \quad \sin\gamma = 0 \tag{1.4}$$

The second equation yields two roots, $(\gamma_1 = 0$ and $\gamma_2 = \pi)$, which have the corresponding two forms of the stationary motions. In both forms all three points, O_1, O and C lie on a single straight line rotating about the O_1-axis at the frequency ω. In the first form $(\gamma_1 = 0)$ the axis of rotation of the rotor O lies between O_1 and C; in the second form $\gamma_2 = \pi)$ the center of mass lies between O_1 and O. Fig.1 depicts the approximate form of

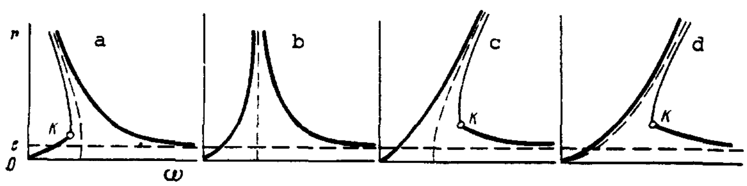

Fig.1

*Prikl.Matem.Mekhan.,Vol.47,No.3,pp.378-384,1983 the possible amplitude-frequency characteristics (a is soft, b is linear, c and d are rigid elastic compliances). The skeleton curve denoted by the dashes

$$\omega = \varkappa(r), \quad \varkappa^2 = F(r)/r \tag{1.5}$$

separates the amplitude-frequency characteristics into two branches, the left branch for the first form ($\gamma_1 = 0$) and the right branch for the second form ($\gamma_2 = \pi$).

Differentiating the first equation of (1.4) with respect to ω, we obtain

$$[F'(r) - \omega^2]\frac{dr}{d\omega} = 2\omega(r \pm e) \tag{1.6}$$

where the upper sign corresponds to the first form of the stationary motion, and the lower sign to the second for (here and henceforth a prime denotes differentiation with respect to r). The equations (1.6) and (1.4) together yield relations determining the point K at which the tangent to the amplitude-frequency characteristics is parallel to the r-axis

$$(r \pm e) F'(r) = F(r), \quad \omega^2 = F'(r) \tag{1.7}$$

For the reaction $F = a\rho^\alpha$, in particular, the point K is given by the coordinates

$$r = \frac{\alpha}{\alpha - 1}e, \quad \omega = \sqrt{a\alpha\left(\frac{\alpha e}{\alpha - 1}\right)^{\alpha - 1}}$$

Setting $\rho = r + z_1$, $\varphi = \omega t - \gamma_j + z_2$ ($j = 1, 2$) and using (1.2) and (1.4), we obtain the following equations of perturbed motion /6/:

$$z_1^{\cdot\cdot} - 2\omega r z_2^{\cdot} + [F'(r) - \omega^2] z_1 = Z_1 \tag{1.8}$$
$$r z_2^{\cdot\cdot} + 2\omega z_1^{\cdot} + r[\varkappa^2(r) \pm \omega^2] z_2 = Z_2$$

where Z_k contains the variations and their derivatives with respect to time, or order higher than first. The equations have a typical gyroscopic structure. Applying the first Tomson–Tet–Chetaev theorem and the conditions of gyroscopic stabilization /7,8/ we study, employing the usual methods, the stability of the zero order solution of the linear part of the system (1.8). Here the specific analytic structure of the reaction $F_0(\rho)$ is of no importance, and we use the general condition (1.1) only. The segments of the amplitude-frequency characteristics (Fig.1), which have the corresponding stationary motions stable in the first approximation, are shown with thick solid lines, while the stationary motions unstable at any Z_k correspond to thin line segments.

2. Let us now take into account the external reaction force $F_c = \mu m v_0$ proportional to the first one, with the axis velocity of the order of O (μ = const). The remaining conditions discussed in Sect.1 remain unchanged. The differential equations of motion now assume the form /6/

$$\rho^{\cdot\cdot} - \rho\varphi^{\cdot 2} - \omega^2 e\cos(\omega t - \varphi) = -F(\rho) - \mu\rho^{\cdot} \tag{2.1}$$
$$\rho\varphi^{\cdot\cdot} + 2\rho^{\cdot}\varphi^{\cdot} - \omega^2 e\sin(\omega t - \varphi) = -\mu\rho\varphi^{\cdot}$$

The constants r and $\gamma = \omega t - \varphi$ satisfy the equations

$$r\omega^2 + e\omega^2\cos\gamma = F(r), \quad e\omega^2\sin\gamma = \mu r\omega \tag{2.2}$$

which define r and γ as functions of ω. The equation can be reduced to the following equivalent form /6/:

$$r^2\{[\varkappa^2(r) - \omega^2]^2 + \mu^2\omega^2\} = e^2\omega^4, \quad \operatorname{tg}\gamma = \frac{\mu\omega}{\varkappa^2(r) - \omega^2} \tag{2.3}$$

From (2.2) or (2.3) it follows that $\gamma \to \pi$ and $r \to e$ as $\omega \to \infty$, which corresponds to the phenomenon of self-centering (which takes place also in the absence of external resistance, see Fig.1). The phenomenon has been studied in detail in /1–3/ for the case of linear reactions $F_0 = c\rho$. The amplitude-frequency characteristics represent a continuous line lying between two open branches corresponding to the case $\mu = 0$. The point A of intersection of the amplitude-frequency characteristics (2.3) with the skeleton curve (1.5) is given by the equations

$$r\mu = \varkappa(r)e, \quad \omega = \frac{\mu}{e}r, \quad \gamma = \frac{\pi}{2} \tag{2.4}$$

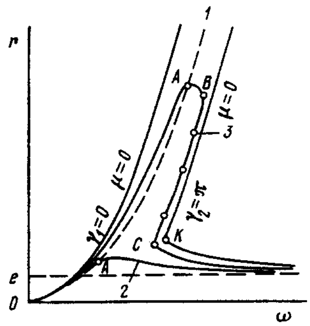

Fig.2

Thus for the reaction $F = a\rho^{\alpha}$ we have, at point A,

$$r = \left(a\frac{e^2}{\mu^2}\right)^{1/(3-\alpha)}, \quad \omega = \frac{\mu}{e}\left(a\frac{e^2}{\mu^2}\right)^{1/(3-\alpha)}, \quad \gamma = \frac{\pi}{2}$$

Differentiating the equations (2.2) with respect to ω and eliminating $d\gamma/d\omega$, we obtain

$$\{[\varkappa^2(r) - \omega^2][F'(r) - \omega^2] + \mu^2\omega^2\}\frac{dr}{d\omega} = \frac{\omega}{r}[2F(r)e\cos\gamma + \mu^2 r^2] \qquad (2.5)$$

and using (2.2) and (2.5) we obtain

$$4e^2F^2(r) - 4\mu^2 r^3 F(r) + \mu^4 r^4 = 0$$

the root of which determines the largest radius r of the precession orbit, with the corresponding values of ω and γ given therefore by (2.2). If the coefficient accompanying $dr/d\omega$

$$a_4 = [\varkappa^2(r) - \omega^2][F'(r) - \omega^2] + \mu^2\omega^2 \qquad (2.6)$$

has no real roots $r = r(\omega)$, then the amplitude-frequency characteristics will have no tangents parallel to the r-axis, ortherwise such tangent exist. To illustrate this, Fig.2 depicts the amplitude-frequency characteristics for the reaction $F = a\rho^{\alpha}$ when $1 < \alpha < 2$. The first case (absence of real roots a_4) corresponds to the characteristics 2, and the second case by the characteristics 3.

In the general case, when the reaction $F(\rho)$ is nonlinear, the first equation of (2.3) has, for certain values of ω, several roots, although its structure clearly implies that for every value of the damping coefficient $\mu = \text{const}$ a small value of the eccentricity $e_0(\mu)$ can always be found such, that when $e < e_0$ then the equation (2.3) will have a single positive root $r = r(\omega)$ of the order $O(e)$. Indeed, when μ is given and ω is arbitrary but fixed, the function

$$\Phi(r) = rf(r) - e\omega^2, \quad f(r) = \sqrt{[\varkappa^2(r) - \omega^2]^2 + \mu^2\omega^2}$$

will depend only on r and the small parameter e, and $f(r) > 0$ for all r, ω and μ not simultaneously all zero. We shall seek a root of the equation $\Phi(r) = 0$, i.e. of (2.3), in the form of a series in powers of e. We find

$$r_1 = \frac{\omega^2}{f(0)}e + Q(e^2)$$

Let us assume that the function $\Phi(r)$ has, at sufficiently small e, more than one root $r(\omega)$. Then we shall have $\Phi(r_*) \leqslant 0$ at the minimum of the function $\Phi(r)$. The value of r_* is found from the equation

$$\Phi'(r_*) = f(r_*) + r_* f'(r_*) = 0$$

which is independent of e. Using $r_* = r_*(\omega)$ obtained, we select a small value $e(\mu, \omega) > 0$ for which $\Phi(r_*) > 0$. This can be done since the function Φ depends continuously on e. When $e = 0$, we have for all $r > 0$ $\Phi(r) > 0$. The contradiction with $\Phi(r_*) \leqslant 0$ shows that for any ω chosen a sufficiently small value $e(\mu, \omega)$ can be found for which the equation (2.3) will have only one positive root $r = r(\omega)$. We can choose as the limiting value of the eccentricity $e_0(\mu)$, $\inf e(\mu, \omega)$ with $\omega \in (0, \infty)$. Sometimes the limiting value $e_0(\mu)$ can be obtained explicitly. Thus, when $F = a\rho^{\alpha}$ $(\alpha > 1)$, the coefficient a_4 becomes

$$a_4 = (ar^{\alpha-1} - \omega^2)(\alpha a r^{\alpha-1} - \omega^2) + \mu^2\omega^2$$

The roots of this quadratic trinomial in $r^{\alpha-1}$ and equation (2.3) together determine the points B and C at which the tangents to the amplitude-frequency characteristics are parallel to the r-axis. Equating the discriminant of this trinomial to zero and using (2.3), we obtain the value of the eccentricity $e_0(\mu)$ at which the points B and C coincide

$$e_0 = \frac{\alpha - 1}{2\alpha}(\alpha + 1)^{1/2}\left[2\frac{\alpha + 1}{\alpha - 1}\frac{\mu^2}{a}\right]^{1/(\alpha-1)}$$

and this implies that when $e < e_0$, then every value of the frequency ω has a corresponding single orbit of cylindrical precession (curve 2 in Fig.2), while $e > e_0$ has three orbits (curve 3).

Setting $\rho = r + z_1$ and $\varphi = \omega t - \gamma + z_2$ (z_k are variations of the coordinates) we obtain from (2.1) the following equations of perturbed motion /6/:

$$z_1^{\cdot\cdot} + \mu z_1^{\cdot} - 2r\omega z_2^{\cdot} + [F'(r) - \omega^2] z_1 - \mu r\omega z_2 = Z_1$$
$$r z_2^{\cdot\cdot} + \mu r z_2^{\cdot} + 2\omega z_1^{\cdot} + \mu\omega z_1 + r[\varkappa^2(r) - \omega^2] z_2 = Z_2 \tag{2.7}$$

where Z_j are terms containing z_k and $z_k^{\cdot}$ of degree higher than the first. The characteristic equation of the system (2.7) reduces to the form

$$\lambda^4 + 2\mu\lambda^3 + [\varkappa^2(r) + F'(r) + 2\omega^2 + \mu^2]\lambda^2 + [\varkappa^2(r) + F'(r) + 2\omega^2]\lambda + [\varkappa^2(r) - \omega^2][F'(r) - \omega^2] + \mu^2\omega^2 = 0 \tag{2.8}$$

By virtue of the condition (1.1), the Hurwitz determinant

$$\Delta_3 = \mu^2 \{[\varkappa^2(r) - F'(r)]^2 + 2[\varkappa^2(r) + F'(r)](4\omega^2 + \mu^2)$$

and all coefficients of (2.8) except the last one, are always positive, therefore the stationary motion will be asymptotically stable in $\rho, \rho^{\cdot}, \varphi$ and $\varphi^{\cdot}$ when $a_4 > 0$, where a_4 coincides with (2.6), and unstable when $a_4 < 0$, neither assertion depending on the higher order terms. The equation $a_4 = 0$ corresponds to the bifurcation points at which the tangents to the amplitude-frequency characteristics are parallel to the r-axis. For this reason we find that for the reaction $F = a\rho^{\alpha}$ the whole characteristics 2 and the segments OB and CD of the characteristics 3 (Fig.2) have the corresponding, asymptotically stable precessions, while the unstable precessions correspond to the segment BC. Fig.3 depicts the character of variation in the orbit radius of the stationary motion when the natural frequency of rotation of the rotor increases (decreases) slowly, for the case $e > e_0$, in the case of a soft (rigid) elastic compliance of the bearings (shown in Fig.3a and 3b respectively). Thus we find, for the rotor model in question, that for any reaction $F_0(\rho)$ satisfying the general conditions (1.1), and under careful balancing when the eccentricity becomes smaller than its critical value e_0, the rapidly rotating rotor will offer a stable performance under the most favorable conditions of self-centering. When $e > e_0$ and the reaction $F_0(\rho)$ is nonlinear, then the pressure N exerted on the bearings can reach, on the ascending branch of the amplitude-frequency characteristics, a consderable magnitude, even when the eccentricity is vanishingly small.

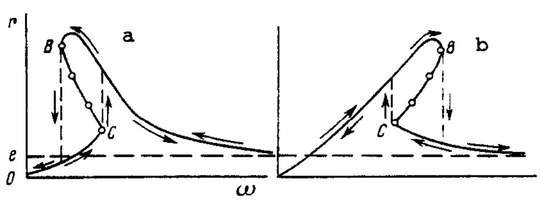

Fig.3

Indeed, when $F_0 = a_0\rho^{3/2}$, the equation of the skeleton curve (1.5) assumes the form $r = \omega^4/a^2 = m^2\omega^4/a_0^2$. Since we have $r(\omega, e) > r(\omega, 0) = m^2\omega^4/a_0^2$ on the segment OA (Fig.2), it follows that for any $e \neq 0$ the total pressure $N = a_0 r^{3/2}$ on the bearings will satisfy the condition

$$N > \frac{m^3}{a_0^2}\omega^6 \tag{2.9}$$

and this implies that the pressure N on the segement OA grows rapidly with increasing frequency ω of natural rotation of the rotor.

3. Assuming, as before, that the rotor is unbalanced ($e \neq 0$), the elastic reaction $F_0(\rho)$ of the bearings satisfies the general conditions (1.1), and the external reaction forces are the same as in Sect.2, we consider the case when the natural rotation of the rotor is realized by an imperfect motor (motor of finite power) generating, on the axis of rotation, the moment $M_0(\psi^{\cdot})$ equal to the difference between the rotational moment $L_0(\psi^{\cdot})$ and the moment of reaction forces $H_0(\psi^{\cdot})$ (ψ is the angle between the segment OC and the fixed x-axis /9/).

The kinetic energy of the rotor is given by the equation

$$T = \frac{m}{2}(\rho^{\cdot 2} + \rho^2\varphi^{\cdot 2}) + me\psi^{\cdot}[-\rho^{\cdot}\sin(\psi - \varphi) + \rho\varphi^{\cdot}\cos(\psi - \varphi)] + \frac{m}{2} i_0^2\psi^{\cdot 2}$$

where i_0 is the radius of inertia of the rotor relative to the axis of rotation O. We use the second Lagrange's method to obtain the equation of motion (the third equation is transformed using the first two equations and represents, in the transformed form, a theorem on the change of angular momentum relative to the center of mass)

$$\rho^{\cdot\cdot} - \rho\varphi^{\cdot 2} - e\psi^{\cdot\cdot}\sin(\psi - \varphi) - e\varphi^{\cdot 2}\cos(\psi - \varphi) = -F(\rho) - \mu\rho^{\cdot} \tag{3.1}$$
$$\rho\varphi^{\cdot\cdot} + 2\rho^{\cdot}\varphi^{\cdot} + e\psi^{\cdot\cdot}\cos(\psi - \varphi) - e\varphi^{\cdot 2}\sin(\psi - \varphi) = -\mu\rho\varphi^{\cdot}$$
$$i_c^2\psi^{\cdot\cdot} + [F(\rho) + \mu\rho^{\cdot}]\, e\sin(\psi - \varphi) - \mu e\rho\varphi^{\cdot}\cos(\psi - \varphi) = M(\psi^{\cdot})$$
$$(i_c^2 = i_0^2 - e^2,\ M(\psi^{\cdot}) = M_0(\psi^{\cdot})/m)$$

This shows that the stationary motions can be realized only under the condition that

$$\rho = r = \text{const}, \quad \psi - \varphi = \gamma = \text{const}, \quad \psi^{\cdot} = \varphi^{\cdot} = \omega = \text{const} \tag{3.2}$$

The constants r, γ, ω and the moment $M(\omega)$ in the stationary motion are connected by the equations

$$r\omega^2 + e\omega^2 \cos\gamma = F(r), \quad e\omega^2 \sin\gamma = \mu r\omega, \quad M(\omega) = \mu r^2 \omega \tag{3.3}$$

The first two equations coincide with (2.2), the latter establishing the dependence of r and γ on ω in the case of a perfect motor. The last equation shows that when the motor is imperfect and the reaction forces $(\mu \neq 0)$ are present, then the angular velocity of the rotor in the steady state motion depends on the radius orbit of the cylindrical precession the condition well known in the case of oscillating systems with nonlinear motor /9/.

Setting $\psi - \varphi = \theta$, $\rho = r + z_1$, $\theta = \gamma + z_2$, $\psi^{\cdot} = \omega + z_3$, we obtain from (3.1) the following equations of perturbed motion:

$$\begin{aligned}
&z_1^{\cdot\cdot} + \mu z_1^{\cdot} + [F'(r) - \omega^2] z_1 + 2r\omega z_2^{\cdot} + e\omega^2 \sin\gamma z_2 - \\
&\quad e \sin\gamma z_3^{\cdot} - 2\omega(r + e\cos\gamma) z_3 = Z_1 \\
&-2\omega z_1^{\cdot} - \mu\omega z_1 + r z_2^{\cdot\cdot} + \mu r z_2^{\cdot} + e\omega^2 \cos\gamma z_2 - (r + \\
&\quad e\cos\gamma) z_3^{\cdot} + \mu r z_3 = Z_2 \\
&\mu e \sin\gamma z_1^{\cdot} + e[F'(r)\sin\gamma - \mu\omega\cos\gamma] z_1 + \mu e r \cos\gamma z_2^{\cdot} + \\
&\quad e[F(r)\cos\gamma + \mu r\omega \sin\gamma] z_2 + i_c^2 z_3^{\cdot} + (k - \mu e r\cos\gamma) z_3 = Z_3
\end{aligned} \tag{3.4}$$

in which $\sin\gamma$ and $\cos\gamma$ are easily eliminated with help of the first two equations of (3.3). The coefficient

$$k = -(dM/d\psi^{\cdot})_{\psi^{\cdot}=\omega} > 0 \tag{3.5}$$

(For a motor intended to maintain a constant frequency of rotation, the derivative $dM/d\psi^{\cdot}$ must be negative). When the reaction forces are absent $(\mu = 0)$, the amplitude-frequency characteristics and two forms of cylindrical precession do not, according to (1.4) and (3.3), depend on the type of the motor, although the stability depends significantly on the fact whether a perfect or imperfect motor is used. Indeed, let us write the characteristic equation for the linear part of the system (3.4), and find the Hurwitz determinants

$$\Delta_2 = \pm ker\omega^2 (r \pm e)^2, \quad \Delta_4 = \pm 4k^2 e^3 r\omega^6 (r \pm e)^2$$

where the upper signs correspond to the first form $(\gamma_1 = 0)$ and the lower signs to the second form $(\gamma_2 = \pi)$ of cylindrical precessions. From this we find that the second form of the stationary motion is always unstable and cannot be therefore realized. Using the condition (1.1) and equations (1.6), we find the asymptotically stable in $\rho, \rho^{\cdot}, \theta, \theta^{\cdot}$ and $\psi^{\cdot}$ precessions of the first form (shown in Fig.4 with thick lines). Comparison of Figs.1 and 4 shows that at $\mu = 0$ the application of an imperfect motor reduces appreciably the regions of stable precessions, but simple stability which occurs with the perfect motor is replaced by the asymptotic stability. When $\mu = 0$, the above conclusions hold for any nonlinear terms Z_k, any eccentricity e, any reaction $F_0(\rho)$ satisfying the general conditions (1.1) and for any imperfect motor for which the inequality (3.5) holds.

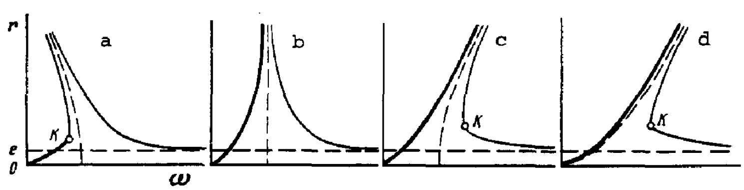

Fig.4

Analyzing the equations (3.4) we can establish the conditions which must be satisfied by the motor in order for the stationary motion of the rotor axis to be asymptotically stable at $\mu \neq 0$ over the whole amplitude-frequency characteristics. Clearly, for this to occur we must demand, before anything else, that the inequality $e < e_0$ holds. Thus, a careful balancing of the rotor appears to be the necessary condition for the reliability of the performance. Finally, the condition $e < e_0$ cannot always be realized in practice, therefore when the bearing reactions are nonlinear, the latter can be successfully placed, in many cases, in linear elastic assemblies eliminating the apperance of the reactions N of the type (2.9).

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